Local well-posedness and global well-posedness of two-phase flows : compressible-compressible case (Mathematical Analysis of Viscous Incompressible Fluid)

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Local well-posedness and global well-posedness of two-phase flows: compressible-compressible case

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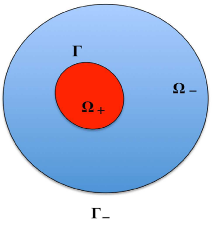
1 Introduction

This article summarizes the joint work with Takayuki Kubo (University of Tsukuba) and Yoshihiro Shibata (Waseda University) [5], [6].

Two phase fluid systems appear in many situation, such as boiling water, carbonated water, etc., accompanied by complicated physics. Lots of efforts have been made to establish physically correct models of two phase fluid systems. Thermodynamically consistent treatment (e.g., satisfaction of the entropy principle) of a two phase flow consisting of multi components is very important in this argument. Thermodynamically consistent modeling often provides complicated constitutive relations and forms of thermodynamical quantities, and therefore the final form of a system of equations would be too complicated to be analyzed mathematically in the current stage. For an overall picture of such modeling, see [1], [2], [3].

Instead of dealing with a physically correct model of two phase system, we consider a simplified problem: *two phase flows in a compact domain consisting of two different compressible barotropic viscous fluids separated by a moving sharp interface under the kinematic conditions, without phase transition, surface tension and body force.* This kind of simplification could be seen as idealization or approximation that is necessary to be done as the first step, especially for possible mathematical analysis of a physically correct full system. Our goal is to perform mathematical analysis on the above simplified system showing local well-posedness [5] and global well-posedness with an additional assumption on the viscosity coefficient and smallness of initial data [6].

We formulate our problem: Let Ω, Ω_+ be connected open subsets of $\mathbb{R}^N$ such that $\Omega_+ \subsetneq \Omega$ and $\Gamma_- := \partial\Omega$, $\Gamma := \partial\Omega_+$ are compact $N-1$ dimensional manifolds. Set $\Omega_- := \Omega \setminus (\Omega_+ \cup \Gamma)$.



At time $t = 0$, the two separated domains $\Omega_\pm$ are occupied by two barotropic viscous fluids with

$$
\begin{array}{rl}
\text{velocity:} & u^0_\pm(\xi) \text{ (satisfying the kinematic condition on } \Gamma, \Gamma_-\text{)},\\
\text{density:} & \bar{\rho}_\pm + \theta^0_\pm(\xi), \quad \bar{\rho}_\pm > 0\text{: reference density (constant)},\\
\text{pressure:} & P_\pm(\bar{\rho}_\pm + \theta^0_\pm(\xi)), \;\; P_\pm \in C^\infty, \;\; P'_\pm > 0,\\
\text{stress tensor:} & S_\pm(u^0_\pm) := 2\mu^1_\pm D(u^0_\pm) + \mu^2_\pm(\nabla \cdot u^0_\pm)I, \;\; D(v) := \dfrac{\nabla v + {}^T\nabla v}{2},\\
 & \mu^1_\pm > 0, \;\; \mu^1_\pm + \mu^2_\pm > 0 \text{ in the case of local well-posedness},\\
 & \mu^1_\pm > 0, \;\; \dfrac{2}{N}\mu^1_\pm + \mu^2_\pm > 0 \text{ in the case of local well-posedness},
\end{array}
$$

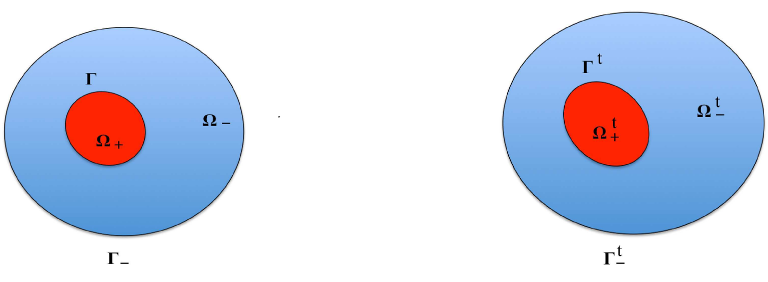
where I is the identity matrix. The domains $\Omega_\pm$ and their boundaries Γ, Γ_- evolve as a family of Lagrangian fluid parcels, where each fluid parcel starting at $\xi \in \Omega_+ \cup \Omega_-$ has the velocity $u(\xi, t)$ and density $\rho^L(\xi, t)$. Let X^t be the flow of fluid parcels,

$$
X^t : \xi \mapsto x(\xi, t) := \xi + \int_0^t u(\xi, s)ds, \;\; \xi \in \Omega_+ \cup \Omega_-, \;\; t \geq 0,
$$

where $x(\xi, t)$ stands for the position at time t of the fluid parcel starting at ξ. Since mass transfer is not taken into account, namely the family of fluid parcel starting from Ω_+ (resp. Ω_-) forms a moving domain with the kinematic condition

$$
\Omega^t_+ := X^t(\Omega_+) \;\; (\text{resp. } \Omega^t_- := X^t(\Omega_-)),
$$

we have the sharp moving interface defined as $\Gamma^t := \partial\Omega^t_+$ that separates the two fluids. The outer boundary of Ω^t_- is denoted by $\Gamma^t_- := \partial\Omega^t_- \setminus \Gamma^t$.



We introduce the notation $u_\pm := u|_{\Omega_\pm}$ and $\rho^L_\pm := \rho^L|_{\Omega_\pm}$. Suppose that $u_\pm$ are smoothly obtained and that $|\int_0^t \nabla u_\pm(\xi, s)ds|$ are small enough for $t \in [0, T]$. Then the Lagrangian transformation can be defined to be a diffeomorphism from $\Omega^t_\pm$ onto $\Omega_\pm$ for each $t \in [0, T]$:

$$
\begin{aligned}
&(X^t)^{-1} : \Omega^t_\pm \ni x \mapsto \xi \in \Omega_\pm, \;\; X^t(\xi) = x \Leftrightarrow \xi = (X^t)^{-1}(x),\\
&D_\xi X^t(\xi) = I + \int_0^t \nabla u_\pm(\xi, s)ds, \;\; \xi \in \Omega_\pm\text{: invertible},\\
&D_x((X^t)^{-1})(x) = (D_\xi X^t(\xi))^{-1} = I + V_0\Big(\int_0^t \nabla u_\pm(\xi, s)ds\Big) \text{ with } X^t(\xi) = x,\\
&V_0(\omega) \to 0(\omega \to 0).
\end{aligned}
$$

We remark that for functions $f(x,t), g(\xi,t)$ such that $f(X^t(\xi),t) \equiv g(\xi,t)$, we have

$$\nabla_x f(X^t(\xi),t) = \nabla_\xi g(\xi,t) + \nabla_\xi g(\xi,t) V_0\Big(\int_0^t \nabla u_\pm(\xi,s)ds\Big).$$

Through the Lagrangian transformation, the free boundary problem is transformed to the fixed boundary problem defined on $\Omega_\pm \cup \Gamma \cup \Gamma_-$. This is the standard approach to free boundary problems without mass transfer across an interface. The governing equations are derived by continuum mechanics in the Eulerian description. The Eulerian fluid field $(v_\pm(x,t), \rho_\pm(x,t), p_\pm(x,t))$ defined for $x \in \Omega_\pm^t$ is given by

$$\begin{aligned} &v_\pm(x,t) := u_\pm((X^t)^{-1}(x),t), \qquad && \rho_\pm(x,t) := \rho_\pm^L((X^t)^{-1}(x),t), \\ &p_\pm(x,t) := P_\pm(\rho_\pm^L((X^t)^{-1}(x),t)). && \end{aligned}$$

Let n^t, n_-^t be the unit outer normal of Γ^t, Γ_-^t respectively. The following are the equations of motion of our system in the Eulerian description: For $t > 0$,

$$(E)\begin{cases} \partial_t \rho_\pm + \nabla\cdot(\rho_\pm v_\pm) = 0 & \text{in } \Omega_\pm^t, \\ \rho_\pm\{\partial_t v_\pm + (v_\pm\cdot\nabla)v_\pm\} + \nabla p_\pm - \mathrm{Div} S_\pm(v_\pm) = 0 & \text{in } \Omega_\pm^t, \\ \lim\limits_{\varepsilon\to 0+} v_-(x+\varepsilon n^t,t) = \lim\limits_{\varepsilon\to 0+} v_+(x-\varepsilon n^t,t) & \text{on } \Gamma^t, \\ \lim\limits_{\varepsilon\to 0+}\{S_-(v_-(x+\varepsilon n^t,t)) - p_-(x+\varepsilon n^t,t)I\}n^t & \\ \quad -\lim\limits_{\varepsilon\to 0+}\{S_+(v_+(x-\varepsilon n^t,t)) - p_+(x-\varepsilon n^t,t)I\}n^t = 0 & \text{on } \Gamma^t, \\ \lim\limits_{\varepsilon\to 0+}\{S_-(v_-(x-\varepsilon n_-^t,t)) - p_-(x-\varepsilon n_-^t,t)I\}n_-^t & \\ \qquad\qquad = -P_-(\bar\rho_-)n_-^t & \text{on } \Gamma_-^t, \\ \rho_\pm(x,0) = \bar\rho_\pm + \theta_\pm^0(x),\ \ v_\pm(x,0) = u_\pm^0(x) & \text{in } \Omega_\pm. \end{cases}$$

The first equation is mass balance; the second one is momentum balance; the third one is continuity of velocity across the interface, which means that fluid parcels never cross the interface (kinematic condition); the fourth one is stress balance across the inner interface; the fifth one is stress balance across the outer interface, where the external pressure is assumed to be the reference pressure of the fluid in Ω_-; the sixth one is the initial condition. The system (E) is transformed into the following problem in the fixed domains by the Lagrangian transformation: Let $\theta_\pm(\xi,t) := \rho_\pm^L(\xi,t) - (\bar\rho_\pm + \theta_\pm^0(\xi))$ with given $\theta_\pm^0$ and n, n_- be the unit outer normals of Γ, Γ_-. Then we have for $t > 0$,

$$(L)\begin{cases} \partial_t\theta_\pm + (\bar\rho_\pm + \theta_\pm^0)\nabla\cdot u_\pm = \mathcal{N}_\pm^1 & \text{in } \Omega_\pm, \\ (\bar\rho_\pm + \theta_\pm^0)\partial_t u_\pm + \nabla\{P'_\pm(\bar\rho_\pm + \theta_\pm^0)\theta_\pm\} & \\ \qquad\qquad -\mathrm{Div} S_\pm(u_\pm) = g_\pm^0 + \mathcal{N}_\pm^2 & \text{in } \Omega_\pm, \\ \lim\limits_{\varepsilon\to 0+} u_-(\xi+\varepsilon n,t) = \lim\limits_{\varepsilon\to 0+} u_+(\xi-\varepsilon n,t) & \text{on } \Gamma, \\ \lim\limits_{\varepsilon\to 0+}\{S_-(u_-(\xi+\varepsilon n,t)) - P'_-(\bar\rho_- + \theta_-^0(\xi+\varepsilon n))\theta_-(\xi+\varepsilon n,t)I\}n & \\ -\lim\limits_{\varepsilon\to 0+}\{S_+(u_+(\xi-\varepsilon n,t)) - P'_+(\bar\rho_+ + \theta_+^0(\xi-\varepsilon n))\theta_+(\xi-\varepsilon n,t)I\}n & \\ \qquad\qquad = h^0 + \mathcal{N}^3 & \text{on } \Gamma, \\ \lim\limits_{\varepsilon\to 0+}\{S_-(u_-(\xi-\varepsilon n_-,t)) - P'_-(\bar\rho_- + \theta_-^0(\xi-\varepsilon n_-))\theta_-(\xi-\varepsilon n_-,t)I\}n_- & \\ \qquad\qquad = h_-^0 + \mathcal{N}^4 & \text{on } \Gamma_-, \\ \theta_\pm(\xi,0) = 0,\ \ u_\pm(\xi,0) = u_\pm^0(\xi) & \text{in } \Omega_\pm, \end{cases}$$

where

$$\mathcal{N}^i_\pm, \mathcal{N}^j = [\text{nonlinear terms of } \theta_\pm, \tfrac{\partial\theta_\pm}{\partial\xi_k}, \tfrac{\partial u_\pm}{\partial t}, \tfrac{\partial u_\pm}{\partial\xi_k}, \tfrac{\partial^2 u_\pm}{\partial\xi_k\xi_l}, \textstyle\int_0^t \nabla u_\pm ds],$$
$$g^0_\pm, h^0, h^0_- = [\text{terms of } \theta^0\pm, \nabla\theta^0_\pm].$$

We prove well-posedness of (L) in a suitable function space so that the invertible Lagrangian transformation is well-defined, yielding well-posedness of (E) as well.

2 Result and idea of proof: local well-posedness

Let $W^m_q(D)$ be the usual Sobolev space of functions defined on D, where $W^0_q(D) := L_q(D)$ is the Lebesgue space. Let $W^m_p((0,T),X)$ be the usual Sobolev space of X-valued functions defined on the interval $(0,T)$. For $0 < \theta < 1$ and $l = 1,2$, let $B^{l,\theta}_{q,p}(D)$ denote the real interpolation space defined by $B^{l,\theta}_{q,p}(D) := (L_q(D), W^l_q(D))_{\theta,p}$ with the real interpolation functor $(\cdot,\cdot)_{\theta,p}$. We say that Γ, Γ_- are W^m_q-manifolds, if they are manifolds with charts of the W^m_q-class.

Theorem 2.1. *Let $N \geq 2$, $2 < p < \infty$, $N < q < \infty$ and Γ, Γ_- be $W^{2-1/q}_q$-manifolds. Let initial data $(\theta^0_\pm, u^0_\pm)$ satisfy $\theta^0_\pm \in W^1_q(\Omega_\pm)$, $u^0_\pm \in (B^{2,1-1/p}_{q,p}(\Omega_\pm))^N$. For each $R > 0$, there exists $T = T(R) > 0$ such that if initial data satisfies*

- $\| \theta^0_\pm \|_{W^1_q(\Omega_\pm)} + \| u^0_\pm \|_{B^{2,1-1/p}_{q,p}(\Omega_\pm)} \leq R$,
- *compatibility conditions from (E),*
- $-\bar{\rho}_\pm/2 \leq \theta^0_\pm \leq \bar{\rho}_\pm/2$,

then (L) admits the unique solution $(\theta_\pm, u_\pm)$ as

$$\theta_\pm \in W^1_p((0,T), W^1_q(\Omega_\pm)),$$
$$u_\pm \in \left(W^1_p((0,T), L_q(\Omega_\pm)) \cap L_p((0,T), W^2_q(\Omega_\pm))\right)^N.$$

Furthermore, the solution $(\theta_\pm, u_\pm)$ yields the invertible Lagrangian transformation and hence the unique solution $(\rho_\pm, v_\pm)$ to (E).

Note that [7], [8] showed similar results in the Hölder space. Theorem 2.1 is proved in the standard framework of the contraction mapping principle, where closed estimates required by this argument is obtained through the maximal L_p-L_q regularity theory of the inhomogeneous linear problems:

$$
(L)_l \begin{cases}
\partial_t\theta_\pm + \gamma^1_\pm(\xi)\nabla\cdot u_\pm = f_\pm(\xi,t) & \text{in } \Omega_\pm\times(0,T),\\
\gamma^0_\pm(\xi)\partial_t u_\pm + \nabla\{\gamma^2_\pm(\xi)\theta_\pm\} - \mathrm{Div}S_\pm(u_\pm) = g_\pm(\xi,t) & \text{in } \Omega_\pm\times(0,T),\\
\lim\limits_{\varepsilon\to 0+} u_-(\xi+\varepsilon n,t) = \lim\limits_{\varepsilon\to 0+} u_+(\xi-\varepsilon n,t) & \text{on } \Gamma\times(0,T),\\
\lim\limits_{\varepsilon\to 0+}\{S_-(u_-(\xi+\varepsilon n,t)) - \gamma^2_-(\xi+\varepsilon n)\theta_-(\xi+\varepsilon n,t)I\}n & \\
\quad - \lim\limits_{\varepsilon\to 0+}\{S_+(u_+(\xi-\varepsilon n,t)) - \gamma^2_+(\xi-\varepsilon n)\theta_+(\xi-\varepsilon n,t)I\}n & \\
\qquad = h(\xi,t) & \text{on } \Gamma\times(0,T),\\
\lim\limits_{\varepsilon\to 0+}\{S_-(u_-(\xi-\varepsilon n_-,t)) - \gamma^2_-(\xi-\varepsilon n)\theta_-(\xi-\varepsilon n_-,t)I\}n_- & \\
\qquad = h_-(\xi,t) & \text{on } \Gamma_-\times(0,T),\\
\theta_\pm(\xi,0)=0,\ \ u_\pm(\xi,0)=u^0_\pm(\xi) & \text{in } \Omega_\pm.
\end{cases}
$$

Here $\gamma^i_\pm, f_\pm, g_\pm, h, h_-$ are given functions. In order to state the two key facts on $(\mathrm{L})_l$, we introduce several function spaces:

$W^l_{p,\gamma}((0,\infty),X) := \{f(t)\in L^{loc}_p((0,\infty),X))\,|\,e^{-\gamma t}\partial^i_t f(t)\in L_p((0,\infty),X),\ i=0,\dots,l\}$,
$L_{p,\gamma}((0,\infty),X) := W^0_{p,\gamma}((0,\infty),X)$,
$H^s_{p,\gamma}(\mathbb{R},X) := \{f(t)\in L_p(\mathbb{R},X)\,|\,e^{-\gamma' t}\Lambda^s_{\gamma'}f(t)\in L_p(\mathbb{R},X)$ for any $\gamma'\ge\gamma\}$ with
$\Lambda^s_\gamma f := \mathcal{L}^{-1}\big[\lambda^s\mathcal{L}[f](\lambda)\big]$, $\mathcal{L}$: Fourier-Laplace transform (Bessel potential space),
$H^s_{p,\gamma}((0,\infty),X) := \{f|_{t>0}\ |\ f\in H^s_{p,\gamma}(\mathbb{R},X)\}$.

The key facts are:

Lemma 2.2. *Let $f_\pm=0, g_\pm=0, h=0, h_-=0$. Then there exist $\gamma_1>0$, $C>0$ such that, for any*

$$\theta^0_\pm\in W^1_q(\Omega_\pm),\ \ u^0_\pm\in(B^{2,1-1/p}_{q,p}(\Omega_\pm))^N$$

satisfying the compatibility conditions from $(L)_l$, we have the unique solution $(\theta_\pm,u_\pm)$ to $(L)_l$ as

$$\begin{aligned}&\theta_\pm\in W^1_{p,\gamma_1}((0,\infty),W^1_q(\Omega_\pm)),\\ &u_\pm\in L_{p,\gamma_1}((0,\infty),(W^2_q(\Omega_\pm))^N)\cap W^1_{p,\gamma_1}((0,\infty),(L_q(\Omega_\pm))^N)\end{aligned}$$

with the estimate

$$\begin{aligned}&\|\,e^{-\gamma t}(\partial_t\theta_\pm,\gamma\theta_\pm)\,\|_{L_p((0,\infty),W^1_q(\Omega_\pm))} + \|\,e^{-\gamma t}(\partial_t u_\pm,\gamma u_\pm)\,\|_{L_p((0,\infty),L_q(\Omega_\pm))}\\ &\quad+\|\,e^{-\gamma t}u_\pm\,\|_{L_p((0,\infty),W^2_q(\Omega_\pm))}\le C\{\|\,\theta^0_\pm\,\|_{W^1_q(\Omega_\pm)}+\|\,u^0_\pm\,\|_{B^{2,1-1/p}_{q,p}(\Omega_\pm)}\}\end{aligned}$$

for any $\gamma\ge\gamma_1$.

Lemma 2.3. *Let $\theta^0_\pm=0$, $u^0_\pm=0$. Then there exists $\gamma_2>0$, $C>0$ such that, for each*

$f_\pm\in L_{p,\gamma_2}((0,\infty),W^1_q(\Omega_\pm))$, $g_\pm\in L_{p,\gamma_2}((0,\infty),(L_q(\Omega_\pm))^N)$,
$h,h_-\in L_{p,\gamma_2}((0,\infty),(W^1_q(\Omega))^N)\cap H^{1/2}_{p,\gamma_2}((0,\infty),(L_q(\Omega))^N)$,

$(L)_l$ admits the unique solution $(\theta_\pm,u_\pm)$ to $(L)_l$ as

$$\begin{aligned}&\theta_\pm\in W^1_{p,\gamma_2}((0,\infty),W^1_q(\Omega_\pm)),\\ &u_\pm\in L_{p,\gamma_2}((0,\infty),(W^2_q(\Omega_\pm))^N)\cap W^1_{p,\gamma_2}((0,\infty),(L_q(\Omega_\pm))^N),\end{aligned}$$

with the estimate

$$
\begin{aligned}
&\| e^{-\gamma t}(\partial_t\theta_\pm, \gamma\theta_\pm) \|_{L_p((0,\infty),W_q^1(\Omega_\pm))} \\
&\quad + \| e^{-\gamma t}(\partial_t u_\pm, \gamma u_\pm, \Lambda_\gamma^{1/2}\nabla u_\pm, \nabla^2 u_\pm) \|_{L_p((0,\infty),L_q(\Omega_\pm))} \\
&\qquad\qquad \leq C\{\| e^{-\gamma t}\partial_t f_\pm \|_{L_p((0,\infty),W_q^1(\Omega_\pm))} + \| e^{-\gamma t} g_\pm \|_{L_p((0,\infty),(L_q(\Omega_\pm))^N)} \\
&\qquad\qquad\qquad + \| e^{-\gamma t}(\Lambda_\gamma^{1/2}h, \nabla h, \Lambda_\gamma^{1/2}h_-, \nabla h_-) \|_{L_p((0,\infty),L_q(\Omega_\pm))}\}
\end{aligned}
$$

for any $\gamma \geq \gamma_2$.

These two lemmas are proved by Weis's operator valued Fourier multiplier theorem and $\mathcal{R}$-boundedness of the solution operator to the resolvent problem derived from $(\mathrm{L})_l$. For this purpose we consider the generalized resolvent problem:

$$
(L)_r \begin{cases}
\lambda\theta_\pm + \gamma_\pm^1(\xi)\nabla\cdot u_\pm = f_\pm(\xi,\lambda) & \text{in } \Omega_\pm, \\
\gamma_\pm^0(\xi)\lambda u_\pm + \nabla\{\gamma_\pm^2(\xi)\theta_\pm\} - \mathrm{Div} S_\pm(u_\pm) = g_\pm(\xi,\lambda) & \text{in } \Omega_\pm, \\
\lim_{\varepsilon\to 0+} u_-(\xi+\varepsilon n,\lambda) = \lim_{\varepsilon\to 0+} u_+(\xi-\varepsilon n,\lambda) & \text{on } \Gamma, \\
\lim_{\varepsilon\to 0+}\{S_-(u_-(\xi+\varepsilon n,\lambda)) - \gamma_-^2(\xi+\varepsilon n)\theta_-(\xi+\varepsilon n,\lambda)I\}n & \\
-\lim_{\varepsilon\to 0+}\{S_+(u_+(\xi-\varepsilon n,\lambda)) - \gamma_+^2(\xi-\varepsilon n)\theta_+(\xi-\varepsilon n,\lambda)I\}n & \\
\qquad = h(\xi,\lambda) & \text{on } \Gamma, \\
\lim_{\varepsilon\to 0+}\{S_-(u_-(\xi-\varepsilon n_-,\lambda)) - \gamma_-^2(\xi-\varepsilon n)\theta_-(\xi-\varepsilon n_-,\lambda)I\}n_- & \\
\qquad = h_-(\xi,\lambda) & \text{on } \Gamma_-.
\end{cases}
$$

The unknown functions $\theta_\pm$ are removed through the first equation to obtain the following problem:

$$
(\tilde{L})_r \begin{cases}
\gamma_\pm^0(\xi)\lambda u_\pm - \lambda^{-1}\nabla\{\gamma_\pm^1(\xi)\gamma_\pm^2(\xi)\nabla\cdot u_\pm\} & \\
\qquad -\mathrm{Div} S_\pm(u_\pm) = \tilde{g}_\pm(\xi,\lambda) & \text{in } \Omega_\pm, \\
\lim_{\varepsilon\to 0+} u_-(\xi+\varepsilon n,\lambda) = \lim_{\varepsilon\to 0+} u_+(\xi-\varepsilon n,\lambda) & \text{on } \Gamma, \\
\lim_{\varepsilon\to 0+}\{S_-(u_-(\xi+\varepsilon n,\lambda)) + \lambda^{-1}\gamma_-^1(\xi+\varepsilon n)\gamma_-^2(\xi+\varepsilon n)\nabla\cdot u_-(\xi+\varepsilon n)I\}n & \\
-\lim_{\varepsilon\to 0+}\{S_+(u_+(\xi-\varepsilon n,\lambda)) + \lambda^{-1}\gamma_+^1(\xi-\varepsilon n)\gamma_+^2(\xi-\varepsilon n)\nabla\cdot u_+(\xi-\varepsilon n)I\}n & \\
\qquad = \tilde{h}(\xi,\lambda) & \text{on } \Gamma, \\
\lim_{\varepsilon\to 0+}\{S_-(u_-(\xi-\varepsilon n,\lambda)) + \lambda^{-1}\gamma_-^1(\xi-\varepsilon n_-)\gamma_-^2(\xi-\varepsilon n_-)\nabla\cdot u_-(\xi-\varepsilon n_-)I\}n_- & \\
\qquad = \tilde{h}_-(\xi,\lambda) & \text{on } \Gamma_-,
\end{cases}
$$

$(\tilde{L})_r$ is reduced to the superposition of solutions to whole or half-space problems by the localization technique:

- Note that $\bar{\Omega} = \Omega_+ \cup \Omega_- \cup \Gamma \cup \Gamma_- \subset \mathbb{R}^N$ is compact.
- Consider a (fine) covering $\{B_i\}_{i\in I}$ of $\bar{\Omega}$, where B_i are open balls.
- Take diffeomorphisms $\Phi_i : \mathbb{R}^N \to \Phi(\mathbb{R}^N)$ such that
  $\Phi_i(\mathbb{R}^N) \supset B_i$, $\Phi_i(\mathbb{R}_0^N) \cup B_i \subset B_i \cap (\Gamma \cup \Gamma_-)$,
  $\Phi_i(\mathbb{R}_{+/-}^N) =$ inside/outside across Γ, Γ_-, if $\Phi_i(\mathbb{R}^N), \Gamma, \Gamma_-$ intersect each other,
  $\mathbb{R}_0^N := \mathbb{R}^N|_{x_N=0}$, $\mathbb{R}_+^N := \mathbb{R}^N|_{x_N>0}$, $\mathbb{R}_-^N := \mathbb{R}^N|_{x_N<0}$.

- Take $[0,1]$-valued smooth indicator functions χ_i, $\tilde{\chi}_i$ on $\mathbb{R}^N$ such that

$$\mathrm{supp}\chi_i \subset \mathrm{supp}\tilde{\chi}_i \subset B_i,\ \tilde{\chi}_i \equiv 1 \text{ on } \mathrm{supp}\chi_i,\ \sum_{i\in I}\chi_i \equiv 1 \text{ on } \Omega.$$

- For example, for B_i such that $B_i \cap \Gamma \neq \phi$, we consider

$$(\tilde{L})_{r,i}\begin{cases} (\tilde{\chi}_i\gamma^0_\pm)\lambda u^i_\pm - \lambda^{-1}\nabla\{(\tilde{\chi}_i\gamma^1_\pm)(\tilde{\chi}_i\gamma^2_\pm)\nabla\cdot u^i_\pm\} \\ \qquad\qquad -\mathrm{Div}S_\pm(u^i_\pm) = (\chi_i\tilde{g}_\pm) & \text{in } \Phi(\mathbb{R}^N_\pm), \\ \lim\limits_{\varepsilon\to 0+} u^i_-(\xi+\varepsilon n,\lambda) = \lim\limits_{\varepsilon\to 0+} u^i_+(\xi-\varepsilon n,\lambda) & \text{on } \Phi(\mathbb{R}^N_0), \\ \lim\limits_{\varepsilon\to 0+}\{S_-(u^i_-(\xi+\varepsilon n,\lambda)) + \lambda^{-1}(\tilde{\chi}_i\gamma^1_-)(\tilde{\chi}_i\gamma^2_-)\nabla\cdot u^i_-(\xi+\varepsilon n)I\}n \\ -\lim\limits_{\varepsilon\to 0+}\{S_+(u^i_+(\xi-\varepsilon n,\lambda)) + \lambda^{-1}(\tilde{\chi}_i\gamma^1_+)(\tilde{\chi}_i\gamma^2_+)\nabla\cdot u^i_+(\xi-\varepsilon n)I\}n \\ \qquad\qquad = (\chi_i\tilde{h}) & \text{on } \Phi(\mathbb{R}^N_0). \end{cases}$$

where $\gamma^0_\pm, \gamma^1_\pm, \gamma^2_\pm$ are extended to be zero outside B_i.

- The other cases than $(\tilde{L})_{r,i}$ are already solved in [4].
- The solutions of $(\tilde{L})_r$ is recovered through $\sum_{i\in I}\chi_i u_i$, where u_i are solutions of $(\tilde{L})_{r,i}$.
- Changing variable by $\xi = \Phi_i(x)$ in $(\tilde{L})_{r,i}$, we obtain the so-called model problem of the form

$$(\tilde{L})_{model}\begin{cases} \gamma^0_\pm\lambda u_\pm + \lambda^{-1}\nabla\{\gamma^1_\pm(\nabla\cdot u_\pm)\} - \mathrm{Div}S_\pm(u_\pm) = g_\pm & \text{in } \mathbb{R}^N_\pm, \\ \lim\limits_{\varepsilon\to 0+} u_-(x+\varepsilon n) - \lim\limits_{\varepsilon\to 0+} u_+(x-\varepsilon n) = k & \text{on } \mathbb{R}^N_0, \\ \lim\limits_{\varepsilon\to 0+}\{S_-(u_-(x+\varepsilon n)) - \lambda^{-1}\gamma^1_-(\nabla\cdot u_-(x+\varepsilon n))I\}n \\ \quad -\lim\limits_{\varepsilon\to 0+}\{S_+(u_+(x-\varepsilon n)) - \lambda^{-1}\gamma^1_+(\nabla\cdot u_+(x-\varepsilon n))I\}n \\ \qquad\qquad = h & \text{on } \mathbb{R}^N_0, \end{cases}$$

where $n = (0,\dots,0,-1)$.

Following the point-wise treatment shown in [4], further reduction of $(\tilde{L})_{model}$ into the following problem with constant coefficients is available:

$$(L)_{model}\begin{cases} \gamma^0_\pm\lambda u_\pm + \delta_\pm\nabla(\nabla\cdot u_\pm) - \mathrm{Div}S_\pm(u_\pm) = 0 & \text{in } \mathbb{R}^N_\pm, \\ \lim\limits_{\varepsilon\to 0+} u_-(x+\varepsilon n) - \lim\limits_{\varepsilon\to 0+} u_+(x-\varepsilon n) = k & \text{on } \mathbb{R}^N_0, \\ \lim\limits_{\varepsilon\to 0+}\{S_-(u_-(x+\varepsilon n)) - \delta_-(\nabla\cdot u_-(x+\varepsilon n))I\}n \\ \quad -\lim\limits_{\varepsilon\to 0+}\{S_+(u_+(x-\varepsilon n)) - \delta_+(\nabla\cdot u_+(x-\varepsilon n))I\}n \\ \qquad\qquad = h & \text{on } \mathbb{R}^N_0. \end{cases}$$

By Fourier-Laplace transformation, solutions to $(L)_{model}$ are represented explicitly. Through analysis of the Lopatinski determinant under the assumption $\mu^1_\pm + \mu^2_\pm > 0$ for the viscosity coefficients, we obtain a $\mathcal{R}$-bounded solution operator defined on

$$\Lambda_{\varepsilon,\lambda_0} = \{\lambda\in\mathbb{C}\setminus\{0\} \,:\, |\arg\lambda| \le \pi-\varepsilon,\ |\lambda|\ge\lambda_0,\cdots\}.$$

We may go back to the original problem with this $\mathcal{R}$-bounded solution operator, to obtain Lemma 2.2 and Lemma 2.3. More details are shown in [5].

3 Result and idea of proof: global well-posedness

Consider the Lagrangian transform of (E) into (L) with $\theta_\pm := \rho^L_\pm - \bar\rho_\pm$ instead of $\theta_\pm := \rho^L_\pm - (\bar\rho_\pm + \theta^0_\pm)$ (we assume smallness of initial data in this section). Then we have

$$(L)\begin{cases} \partial_t\theta_\pm + \bar\rho_\pm \nabla\cdot u_\pm = \mathcal{N}^1_\pm & \text{in } \Omega_\pm, \\ \bar\rho_\pm\partial_t u_\pm + \nabla\{P'_\pm(\bar\rho_\pm)\theta_\pm\} - \mathrm{Div} S_\pm(u_\pm) = \mathcal{N}^2_\pm & \text{in } \Omega_\pm, \\ \lim\limits_{\varepsilon\to 0+} u_-(\xi+\varepsilon n, t) = \lim\limits_{\varepsilon\to 0+} u_+(\xi-\varepsilon n, t) & \text{on } \Gamma, \\ \lim\limits_{\varepsilon\to 0+}\{S_-(u_-(\xi+\varepsilon n,t)) - P'_-(\bar\rho_-)\theta_-(\xi+\varepsilon n,t)I\}n & \\ -\lim\limits_{\varepsilon\to 0+}\{S_+(u_+(\xi-\varepsilon n,t)) - P'_+(\bar\rho_+)\theta_+(\xi-\varepsilon n,t)I\}n = \mathcal{N}^3 & \text{on } \Gamma, \\ \lim\limits_{\varepsilon\to 0+}\{S_-(u_-(\xi-\varepsilon n_-,t)) - P'_-(\bar\rho_-)\theta_-(\xi-\varepsilon n_-,t)I\}n_- = \mathcal{N}^4 & \text{on } \Gamma_-, \\ \theta_\pm(\xi,0) = \theta^0_\pm(\xi), \ \ u_\pm(\xi,0) = u^0_\pm(\xi) & \text{in } \Omega_\pm, \end{cases}$$

where $\mathcal{N}^i$ are nonlinear terms. We put a stronger assumption on the viscosity coefficients as

$$S_\pm(v) = 2\mu^1_\pm D(v) + \mu^2_\pm(\nabla\cdot v)I, \ \ \mu^1_\pm > 0, \ \ \underline{\frac{2}{N}\mu^1_\pm + \mu^2_\pm > 0,}$$

which is still physically standard.

Since we seek for fluid motion tending to zero, i.e., the motion without motion as a rigid body, we assume the orthogonal condition of initial data: Set the rigid space

$$\{v(x) : \Omega \to \mathbb{R}^N \mid \nabla v + {}^T\nabla v \equiv 0\}.$$

It is known that the rigid space has the orthogonal basis

$$\{b_{ij}\}_{i,j=1,2,\dots,N} := \{x_i e_j - x_j e_i\}_{i,j=1,2,\dots,N}.$$

The orthogonal condition of initial data is

$$\sum_\pm ((\bar\rho_\pm + \theta^0_\pm)u^0_\pm, b_{ij})_{L_2(\Omega_\pm)} = 0, \ \ i,j = 1,2,\dots,N.$$

For a fluid field $(v_\pm, \rho_\pm)$ in the Eulerian description, we have conservation of angular momentum:

$$\sum_\pm \int_{\Omega^t_\pm} \rho_\pm(x,t)v_\pm(x,t)\cdot b_{ij}(x)dx = \sum_\pm \int_{\Omega_\pm} \rho_\pm(0,x)v_\pm(0,x)\cdot b_{ij}(x)dx.$$

If initial data satisfies the orthogonal condition, we have for all $t > 0$,

$$\sum_\pm \int_{\Omega^t_\pm} \rho_\pm(x,t)v_\pm(x,t)\cdot b_{ij}(x)dx = 0.$$

By the Lagrangian transformation

$$x = X^t(\xi), \ \ DX^t(\xi) = I + \int_0^t \nabla u_\pm(\xi,s)ds,$$
$$\det DX^t(\xi) = 1 + \tilde V_0(\int_0^t \nabla u_\pm(\xi,s)ds), \ \ \tilde V_0(y) \to 0 \ \ (y \to 0),$$

we have the conservation of angular momentum in the Lagrangian description

$$\sum_{\pm}\int_{\Omega_\pm}(\bar\rho_\pm+\theta_\pm(\xi,t))u_\pm(\xi,t)\cdot b_{ij}(X^t(\xi))|\mathrm{det}DX^t(\xi)|d\xi=0.$$

Therefore, we see that

$$\sum_{\pm}\int_{\Omega_\pm}\bar\rho_\pm u_\pm(\xi,t)\cdot b_{ij}(\xi)d\xi=[\text{quadratic terms of }\theta_\pm,\ u_\pm,\ \tilde V_0],$$

even though it is apparently a first order term.

Theorem 3.1. *Let $N\geq 2$, $2<p<\infty$, $N<q<\infty$ and Γ,Γ_- be $W_q^{2-1/q}$-manifolds. Let initial data $(\theta^0_\pm,u^0_\pm)$ be such that $\theta^0_\pm\in W^1_q(\Omega_\pm)$, $u^0_\pm\in(B^{2,1-1/p}_{q,p}(\Omega\pm))^N$,*

- *compatibility conditions from (E),*
- $-\bar\rho_\pm/2\leq\theta^0_\pm\leq\bar\rho_\pm/2$,
- *orthogonal condition.*

Then there exists $\varepsilon>0$ such that, if initial data satisfies

$$\|\theta^0_\pm\|_{W^1_q(\Omega_\pm)}+\|u^0_\pm\|_{B^{2,1-1/p}_{q,p}(\Omega\pm)}\leq\varepsilon\ \textit{(smallness condition)},$$

(L) admits the unique global solution $(\theta_\pm,u_\pm)$ as

$$\theta_\pm\in W^1_p((0,\infty),W^1_q(\Omega_\pm)),\ u_\pm\in\left(W^1_p((0,\infty),L_q(\Omega_\pm))\cap L_p((0,\infty),W^2_q(\Omega_\pm))\right)^N,$$

possessing the following estimate: for some $\gamma>0$,

$$\sum_{\pm}\{\|e^{\gamma t}\theta_\pm\|_{W^1_p((0,\infty),W^1_q(\Omega_\pm)}+\|e^{\gamma t}u_\pm\|_{L_p((0,\infty),W^2_q(\Omega_\pm))}$$
$$+\|e^{\gamma t}\partial_t u_\pm\|_{L_p((0,\infty),L_q(\Omega_\pm))}\}\leq C\varepsilon\ (C>0\text{: independent of }\varepsilon).$$

This result yields the invertible Lagrangian transformation and therefore the unique global solution $(\rho_\pm,v_\pm)$ to (E).

Theorem 3.1 is proved by extending a time-local solution to any time interval through a priori estimate and showing exponential stability of the analytic semigroup for the linearized problem. For exponential stability of the semigroup, we study the corresponding resolvent problem for $\lambda\in\{\lambda\in\mathbb{C}\,|\,Re\lambda\geq 0,|\lambda|\leq\lambda_0\}$ with the following strategy:

- $\lambda\neq 0\Rightarrow$ existence: Fredholm alternative principle,
  uniqueness: L_2-energy estimate with compactness of $\Omega_\pm$, $\frac{2}{N}\mu^1_\pm+\mu^2_\pm>0$,
- $\lambda=0\Rightarrow$ generalized Cattabriga theorem.

For extension of a local solution $(\theta_\pm, u_\pm)$ to (L) defined on $(0,T]$, we show the following a priori estimate: Set

$$
\begin{aligned}
E_\gamma^T(\theta_\pm, u_\pm) := \sum_\pm \{ \| e^{\gamma t}\theta_\pm \|_{W_p^1((0,T),W_q^1(\Omega_\pm))} + \| e^{\gamma t} u_\pm \|_{W_p^1((0,T),L_q(\Omega_\pm))} \\
+ \| e^{\gamma t} u_\pm \|_{L_p((0,T),W_q^2(\Omega_\pm))} \}, \\
I_\gamma^T(\theta_\pm^0, u_\pm^0, \mathcal{N}_\pm^1, \mathcal{N}_\pm^2, \mathcal{N}^3, \mathcal{N}^4) := \sum_\pm \{ \| \theta_\pm^0 \|_{W_q^1(\Omega_\pm)} + \| u_\pm^0 \|_{B_{q,p}^{2,1-1/p}(\Omega_\pm)} \\
+ \| e^{\gamma t}\mathcal{N}_\pm^1 \|_{L_p((0,T),W_q^1(\Omega_\pm))} \} + \| e^{\gamma t}\mathcal{N}_\pm^2 \|_{L_p((0,T),L_q(\Omega_\pm))} \} \\
+ \| e^{\gamma t}\mathcal{N}^3 \|_{L_p((0,T),W_q^1(\Omega_\pm))} + \| e^{\gamma t}\partial_t\mathcal{N}^3 \|_{L_p((0,T),W_q^{-1}(\Omega_\pm))} \\
+ \| e^{\gamma t}\mathcal{N}^4 \|_{L_p((0,T),W_q^1(\Omega_\pm))} + \| e^{\gamma t}\partial_t\mathcal{N}^4 \|_{L_p((0,T),W_q^{-1}(\Omega_\pm))} .
\end{aligned}
$$

Then, through the linearized problem of (L), we obtain for all $\gamma \in [0, \gamma_1)$,

$$
E_\gamma^T \le C_\gamma I_\gamma^T + \sum_{ij} \left\{ \int_0^T \left(e^{\gamma t} \underline{\left| \sum_\pm \int_{\Omega_\pm} \bar\rho_\pm u_\pm \cdot b_{ij} d\xi \right|} \right)^p ds \right\}^{1/p}.
$$

Note that —— is quadratic with respect to $\theta_\pm$, $u_\pm$, $\tilde V_0(\int_0^T \nabla u_\pm ds)$, through which we obtain $C, \tilde C > 0$ independent of ε and T such that

$$
E_\gamma^T \le C(\varepsilon + (E_\gamma^T)^2) \quad \text{and hence} \quad E_\gamma^T \le \tilde C \varepsilon.
$$

The quantities $\| \theta_\pm(T) \|_{W_q^1(\Omega_\pm)}$, $\| u_\pm(T) \|_{B_{q,p}^{2,1-1/p}(\Omega_\pm)}$ are estimated with $\| \theta_\pm^0 \|_{W_q^1(\Omega_\pm)}$, $\| u_\pm^0 \|_{B_{q,p}^{2,1-1/p}(\Omega_\pm)}$, E_γ^T, and we may continue to solve (L) up to $t = T + T_\varepsilon$. We still have the same estimates for $E_\gamma^{T+T_\varepsilon}$, $\| \theta_\pm(T+T_\varepsilon) \|_{W_q^1(\Omega_\pm)}$, $\| u_\pm(T+T_\varepsilon) \|_{B_{q,p}^{2,1-1/p}(\Omega_\pm)}$ to extend the solution up to $t = T + 2T_\varepsilon$. Thus we obtain the time-global solution.

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